A SPECIAL PROPERTY OF "ALMOST" UNIFORMLY CONTINUOUS FUNCTIONS

HUY BUI

Abstract. This paper shows a special property of "almost" uniformly continuous functions. This property is related to the probability measures.

1. Introduction

The purpose of this paper is to prove that "almost" continuous and finite functions on a Borel set are measurable. In particular, almost uniformly continuous and finite functions on a Borel set are measurable. We first introduce some basic concepts.

Definition 1. A *σ-algebra* of subsets of the set Ω is a non-empty class $\mathcal{A}$ of sets such that

(a) if $A \in \mathcal{A}$, then $A^c = \Omega \backslash A \in \mathcal{A}$,

(b) for any sequence $\{A_i\}_{i\in\mathbb{N}}$ in $\mathcal{A}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

Whence the pair $(\Omega, \mathcal{A})$ is called a *measurable space*.

Definition 2. The mapping μ from σ-algebra $\mathcal{A}$ into $\mathbb{R}$ is called a *measure* if and only if

(a) for all $A \in \mathcal{A}$, then $\mu(A) \geq 0$ and $\mu(\varnothing) = 0$,

(b) if $\{A_i\}_{i\in\mathbb{N}} \subset \mathcal{A}$ such that $A_i \cap A_j = \varnothing$ for all $i \neq j$, then

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

In particular, if $\mathcal{A}$ is a σ-algebra of subsets of the set Ω and $\mu(\Omega) = 1$, then μ is called a *probability measure*. Whence the triple $(\Omega, \mathcal{A}, \mu)$ is called a *probability space*.

Definition 3. Let X be a metric space. The smallest σ-algebra contains all the open sets of X is called a Borel σ-algebra and its elements are called *Borel sets*.

Let X be a metric space and ρ a metric for X. We denotes $\mathcal{S}$ a class of Borel sets in X and $\mathcal{P}$ a class of probability measures on $\mathcal{S}$. The class $\mathcal{S}$ is the smallest σ-algebra generated by the open sets.

Definition 4. A probability measure μ on $(X, \mathcal{S})$ is said to be *regular* if $A \in \mathcal{S}$ and given an $\epsilon > 0$ then there exist a closed set F and an open set G such that $F \subset A \subset G$ and $\mu(G\backslash F) < \epsilon$.

Definition 5. A function f from the metric space X into the set of real numbers is said to be *almost uniformly continuous* on the subset A in X if it is uniformly continuous on each closed subset B of A.

If the function f is continuous on each closed subset B of A, then f is said to be almost continuous on the subset A.

Definition 6. The mapping f from the measurable space $(X, \mathcal{X})$ into the measurable space $(Y, \mathcal{Y})$ is called *measurable* if and only if for every $B \in \mathcal{Y}$, then $f^{-1}(B) \in \mathcal{X}$.

2. Main Results

The main results of this paper are the following theorem and corollary:

Theorem. *Let μ be a probability measure on the class $\mathcal{S}$ of Borel sets in the metric space X. If a function f from X into $\mathbb{R}$ is almost continuous and finite on the set A of $\mathcal{S}$, then the function f is measurable on A. (The topology on $\mathbb{R}$ is the usual topology)*

Proof. First, we show that μ is a regular measure. Let ρ be a metric on space X and denote the distance from x to A by $\rho(x, A)$. In the case that the set A is closed we take $F = A$ and $G = \{x\colon \rho(x, A) < \delta\}$ for some δ. Since the sets G decrease to A as $\delta \downarrow 0$, we need only the class $\mathcal{F}$ of Borel sets with the asserted property in Definition 4 is a σ-algebra.

Assume that, the sets $A_n \in \mathcal{S}$, $n \in \mathbb{Z}_+$ satisfy the asserted properties of the theorem; that is for given $\epsilon > 0$ there are closed sets F_n and open sets G_n such that $F_n \subset A \subset G_n$ and $\mu(G_n \backslash F_n) < \epsilon$. Hence, given sets A_n in $\mathcal{S}$ and for given $\epsilon > 0$, choose closed sets F_n and open sets G_n such that $F_n \subset A \subset G_n$ and $\mu(G_n \backslash F_n) < \frac{\epsilon}{2^{n+1}}$.

Let

$$F = \bigcup_{n=1}^{n_0} F_n \qquad \text{and} \qquad G = \bigcup_{n=1}^{\infty} G_n,$$

where n_0 is chosen so that

$$\mu(\bigcup_{n=1}^{\infty} F_n \backslash F) < \frac{\epsilon}{2}.$$

Since

$$G \backslash F = (\bigcup_{n=1}^{\infty} G_n \backslash \bigcup_{n=1}^{\infty} F_n) \bigcup (\bigcup_{n=1}^{\infty} F_n \backslash F),$$

and

$$\bigcup_{n=1}^{\infty} G_n \backslash \bigcup_{n=1}^{\infty} F_n \subset \bigcup_{n=1}^{\infty} (G_n \backslash F_n).$$

We obtain

$$\begin{aligned}
\mu(G\backslash F) &= \mu[(\bigcup_{n=1}^{\infty} G_n \backslash \bigcup_{n=1}^{\infty} F_n) \bigcup (\bigcup_{n=1}^{\infty} F_n \backslash F)] \\
&\leq \mu(\bigcup_{n=1}^{\infty} G_n \backslash \bigcup_{n=1}^{\infty} F_n) + \mu(\bigcup_{n=1}^{\infty} F_n \backslash F) \\
&\leq \mu[\bigcup_{n=1}^{\infty} (G_n \backslash F_n)] + \frac{\epsilon}{2} \\
&\leq \sum_{n=1}^{\infty} \mu(G_n \backslash F_n) + \frac{\epsilon}{2} \\
&\leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+1}} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Thus $\mathcal{F}$ is closed under the formation of counntable unions and obviously closed under complementation, so $\mathcal{F}$ is a σ-algebra containing the class of all closed sets. Hence $\mathcal{F}$ contains $\mathcal{S}$. This follows that the measure μ is regular.

Next, we demonstrate the remainder part of the theorem. In view of the preceding paragraph, there are exist the closed set F and the open set G such that $F \subset A \subset G$ and $\mu(G\backslash F) < \epsilon$. Hence, for a positive integer n, choose the closed set F_n and the open set G_n such that

$$F_n \subset A \subset G_n \qquad \text{and} \qquad \mu(G_n \backslash F_n) < \frac{1}{n}.$$

Let

$$F = \bigcap_{n=1}^{\infty} (A \backslash F_n).$$

We have

$$\mu(F) \leq \mu(A\backslash F_n) \leq \mu(G_n \backslash F_n) < \frac{1}{n}, \qquad \forall n \in \mathbb{N}^*$$

since $A\backslash F_n \subset G_n \backslash F_n$.
Hence

$$\mu(F) = 0.$$

Since

$$A\backslash F = \bigcup_{n=1}^{\infty} F_n \qquad \text{(by De Morgan laws)},$$

$$\forall \alpha \in \mathbb{R}, \{x \in A\backslash F \colon f(x < \alpha)\} = \bigcup_{n=1}^{\infty} \{x \in F_n \colon f(x) < \alpha\},$$

and the function f is almost uniformly continuous on A, so it is continuous on the closed set F_n, implying the set $\{x \in F_n \colon f(x) < \alpha\}$ measurable. So, the function $f(x)$ is measurable on the set $A \backslash F$ and obviously measurable on the set A because $\mu(F) = 0$. Thus, the theorem is proved. □

Corollary 7. *Let μ be a probability measure on the class $\mathcal{S}$ of Borel sets of the metric space X. If the function f is almost uniformly continuous and finite on the set A of $\mathcal{S}$, then f is measurable on A.*

Proof. Since an almost uniformly continuous function is almost continuous, so applying the theorem above to obtain the result. □

Corollary 8. *If f is almost continuous on the Borel set A in the metric space X and $\mu$is a probability measure on the class $\mathcal{S}$ containing all Borel sets of X, then f is almost surely continuous on A.*

Proof. Since μ is regular, so for each $n \in \mathbb{Z}_+$, there exists a closed set F_n and an open set G_n such that $F_n \subset A \subset G_n$ and $\mu(G_n \backslash F_n) < \frac{1}{n}$. Thus

$$\mu(A \backslash F_n) \leq \mu(G_n \backslash F_n) < \frac{1}{n}.$$

For each $\epsilon > 0$, there exists a positive number n_0 satisfying $\frac{1}{n_0} < \epsilon$. Denote

$$F = \bigcup_{n=1}^{\infty} F_n \in \mathcal{S},$$

so $F \subset A$. We have that f is continuos on F and

$$\mu(A \backslash F) \leq \mu(A \backslash F_0) \leq \frac{1}{n_0} < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, so $\mu(A \backslash F) = 0$. Therefore the theorem is proved. □

References

[1] Richard Dudley, *Real Analysis and Probability*, Wadsworth, Brooks and Cole (1989)
[2] George L. Cain, *Introduction to General Topology*, Pearson (1994)
[3] James R. Munkres, *Topology*, Pearson (2000)